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Optimal Decision Rules for the Triangular E Model of Chance-Constrained Programming

by
Abraham Charnes
Northwestern University
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McLEAN, VIRGINIA

FOREWORD

This paper establishes properties of the optimal decision rules for a particular class of chance-constrained programming problems. The type of problem considered is an n-period model in which each period generates exactly two constraints. One of these constraints couples the decision rule of the jth period to the decision rules of all succeeding periods, while the other is a constraint requiring the decision rule to be nonnegative with at least a specified probability. Necessary and sufficient conditions for optimality are derived and related to results in the calculus of variations.

One application of the mathematical developments presented in this paper is described in RAC-P-12, "Application of Chance-Constrained Programming to Solution of the So-Called 'Savings and Loan Association' Type of Problem."

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Nicholas M. Smith
Head, Advanced Research Department

CONTENTS

Optimal Decision Rules for the Triangular E Model of Chance-Constrained Programming

ABSTRACT

This paper deals with an n-period E model of chance-constrained programming in which each period $j = 1, \ldots, n$ generates exactly one new constraint. It is shown that there are cases in which the problem can be reduced to one of solving n rather simple one-variable nonlinear programming problems.

The results of this paper are illustrated by means of an example giving the solution of a two-period problem of planning for liquidity in a savings and loan association.

1. INTRODUCTION

In a previous paper[1] the authors established certain necessary conditions for decision rules to be optimal for the block triangular n-period E model of chance-constrained programming. This paper is concerned with a more restricted problem than the one considered in RAC-TP-166.[1] In particular an n-period E model is considered in which each period $j = 1, \ldots, n$ generates exactly one new constraint rather than m_j new constraints as in the earlier paper.

This restriction leads to a problem that is easier to handle mathematically than the more general case considered in RAC-TP-166,[1] about which much more can be said. In fact, in certain cases the problem can be reduced to one of solving n rather simple one-variable nonlinear programming problems. Moreover, in the event that each of the random variables involved in the problem has the same distribution, the complete n-stage problem can be reduced to solving one of these simple nonlinear problems.

In addition, two generalizations of the problem in RAC-TP-166[1] are considered here. First, instead of having the ith constraint be of the form

$$P\left(\sum_{j=1}^{i} a_{ij} X_j \le b_i\right) \ge \alpha_i ,$$

which would be the triangular version of the block triangular n-period problem, the ith constraint is allowed to be of the form

$$P\left(\sum_{j=1}^{i} a_{ij} X_j + d_i b_i + \omega_i(b_1, \ldots, b_{i-1}) \le 0\right) \ge \alpha_i ,$$

where d_i is a constant and $\omega_i(b_1, \ldots, b_{i-1})$ is an arbitrary piecewise continuous function of the random variables $b_1, \ldots, b_{i-1}$. Second, the nonnegativity constraints $X_j \ge 0$, $j = 1, \ldots, n$, used in RAC-TP-166[1] are replaced by the more general constraints $P(X_j \ge 0) \ge \beta_j$, $j = 1, \ldots, n$, where β_j is some preassigned probability.

The effect of this second generalization is discussed at length. It is shown that restricting X_j to be nonnegative only $100\beta_j$ percent of the time, rather than all the time as was done in RAC-TP-166[1], greatly increases the mathematical complexity of the problem. In addition the interesting result is derived that when such a constraint is tight, the optimal rule is often discontinuous where it was previously continuous. From other points of view, this result is to be anticipated. For example (S,s) policies,[2] when optimal, in inventory theory have this property, as do many solutions of optimal control problems.[3]

The simplification that results when all the preassigned probabilities are equal to 1 is also illustrated. In this case the optimal decision rule for period i is a piecewise linear function of the decision rules of periods $1, \ldots, i-1$, and the pieces can be easily found.

An example of the application of the results of this paper to a problem in financial planning is contained in Charnes and Kirby,[4] which gave a solution of a two-period problem of planning for liquidity in a savings and loan association. In this model the fact that the optimal decision rule is in general discontinuous is surprising, but it can be explained by economic arguments.

2. STATEMENT OF THE PROBLEM

The problem to be considered is:
maximize

$$\sum_{j=1}^{n} E(c_j X_j)$$

subject to

$$P(a_{11}X_1 + d_1 b_1 + \omega_1 \leq 0) \geq \alpha_1,$$

$$P(a_{21}X_1 + a_{22}X_2 + d_2 b_2 + \omega_2(b_1) \leq 0) \geq \alpha_2,$$

$$P(a_{31}X_1 + a_{32}X_2 + a_{33}X_3 + d_3 b_3 + \omega_3(b_1, b_2) \leq 0) \geq \alpha_3,$$

$$P\left(\sum_{j=1}^{i} a_{ij}X_j + d_i b_i + \omega_i(b_1, \ldots, b_{i-1}) \leq 0\right) \geq \alpha_i,$$

$$P\left(\sum_{j=1}^{n} a_{nj}X_j + d_n b_n + \omega_n(b_1, \ldots, b_{n-1}) \leq 0\right) \geq \alpha_n,$$

$$P(X_j \geq 0) \geq \beta_j, \; j = 1, \ldots, n, \qquad (1)$$

where P and E represent the probability and expectation operators, respectively. The probability and expectation is computed by using the joint distribution of all the random variables involved in the problem.

In Problem 1 the following assumptions are made:

(a) a_{ij}, $i \geq j$, $i, j = 1, \ldots, n$, d_i, c_i, $i = 1, \ldots, n$, and ω_1 are given constants, and $a_{ii} \neq 0$, $d_i \neq 0$ for all i.

(b) α_i, β_j, $i, j = 1, \ldots, n$ are known probabilities. Thus $0 \leq \alpha_i, \beta_j \leq 1$ for all i and j.

(c) the b_i, $i = 1, \ldots n$ are continuous random variables whose joint frequency function $f_n(b_1, \ldots, b_n)$ is known.

(d) X_j, $j = 1, \ldots, n$ is a function of the random variables $b_1, \ldots, b_{j-1}$ but it is not a function of $b_j, \ldots, b_n$. Thus we will solve Eq 1 for $X_j = X_j(b_1, \ldots, b_{j-1})$.

In Sec 5 we will consider the more general problem that arises when we allow $c_1, \ldots, c_n$ to be random variables rather than constants as they are in assumption a. However, since some of the work in Sec 3 does not extend to this case we will for the moment assume that c_j, $j = 1, \ldots, n$ are given constants.

Assumption d is due to our interpretation of the problem. We are going to treat Problem 1 as an n-period, or n-stage, problem in which X_j, the decision rule for the jth period, is selected after all previous decisions $X_1, \ldots, X_{j-1}$ are known and after the values of the random variables of periods 1 to $j-1$ have been observed but before b_j and all random variables and decisions of periods $j+1$ to n have been observed.

In other words, X_1, the first-period decision rule, must be selected before the value of the first-period random variable b_1 is observed. Then, when we have selected X_1 and observed b_1, the second-period decision rule X_2 must be chosen before the value of b_2 is observed. This process continues with X_j depending explicitly on X_i, b_i, $i = 1, \ldots, j-1$ and only implicitly (i.e., through the coupling effect of the constraints) on b_j and X_i, b_i, $i = j+1, \ldots, n$. It is this interpretation that led us to make assumption d.

The set Q_i, $i = 1, \ldots, n$, in i-dimensional Euclidean space is defined as the set of points $(b_1, \ldots, b_i)$ for which $f_i(b_1, \ldots, b_i) > 0$, where $f_i(b_1, \ldots, b_i)$ is the joint frequency function of $b_1, \ldots, b_i$. Since we have, by definition of $f_i(b_1, \ldots, b_i)$, the identity

$$f_i(b_1, \ldots, b_i) = \int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f_n(b_1, \ldots, b_n) \prod_{j=i+1}^{n} db_j,$$

it may be seen by assumption c that $f_i(b_1, \ldots, b_i)$ is a known function. It is important to note that no restriction on Q_i as a bounded set has been made. Thus it may extend to $\pm\infty$ in any, or all, of its i dimensions.

$F_i(.)$ is used to represent the multivariate cumulative distribution function of the random variables $b_1, \ldots, b_i$. We will write

$$F_i(G) = \int \cdots \int_G f_i(b_1, \ldots, b_i)\, db_1 \ldots db_i,$$

where G is any subset of i-dimensional space.

One more restriction must be placed on our problem in order that the differential equations method of the isoperimetric theory of the calculus of variations may be used.

Assume that for each s, $s = 2, \ldots, n$, there exists a set of $s-1$ dimensional rectangles, say $\{A_\ell^{s-1}, \ell \in \Upsilon^{s-1}\}$, where Υ^{s-1} is some indexing set, such that

(i) $\bar{Q}_{s-1} \subset \bigcup_{\ell \in \Upsilon^{s-1}} A_\ell^{s-1}$, where $\bar{Q}_{s-1}$ is the closure of Q_{s-1}.

(ii) $F_{s-1}(A_\ell^{s-1}) > 0$, for all $\ell \in \Upsilon^{s-1}$,

(iii) $F_{s-1}(A_k^{s-1} \cap A_\ell^{s-1}) = 0$, for all $k, \ell \in \Upsilon^{s-1}$ and $k \neq \ell$,

(iv) $f_{s-1}, \omega_s, \bar{f}_s(a_{ss}X_s^* + \sum_{j=1}^{s-1} a_{sj}X_j + \omega_s), X_j, j = 1, \ldots, s-1$ and X_s^* are continuous in A^{s-1} for all $\ell \in \Upsilon^{s-1}$,

(v) X_s^* is of constant sign in A_ℓ^{s-1} for all $\ell \in \Upsilon^{s-1}$,

where $\bar{f}_s(.)$ is the conditional frequency function of b_s given $b_1, \ldots, b_{s-1}$ and X_s^* is an optimal X_s for 1.

i, ii, and iii say that $\{A_\ell^{s-1}, \ell \in \Upsilon^{s-1}\}$ divides $\bar{Q}_{s-1}$ into a set of $s-1$ dimensional rectangles such that the probability that a random point $(b_1, \ldots, b_{s-1})$ in Q_{s-1} is in any one of these rectangles is greater than zero, and the probability that $(b_1, \ldots, b_{s-1})$ is in the intersection of any two rectangles is zero. In both cases the probability is computed using the frequency function

$f_{s-1}(b_1, \ldots, b_{s-1})$. Moreover, using well-known properties of any distribution function (i.e., for discrete or continuous random variables), it is easy to show, using properties ii and iii, that there is at most a countable number of rectangles A_ℓ^{s-1} with $\ell \in \Omega^{s-1}$.

It is always possible to find $\{A_\ell^{s-1}, \ell \in \Omega^{s-1}\}$ with properties i, ii, and iii and such that f_{s-1} is continuous in each A_ℓ^{s-1}. It follows then that an assumption equivalent to iv would be that X_j, $j = 1, \ldots, n$, and $\omega_j(b_1, \ldots, b_{j-1})$, $j = 1, \ldots, n$ are each continuous functions with a countable number of discontinuities. Thus iv does not restrict our problem to any significant degree.

It is important to realize that the set of rectangles $\{A_\ell^{s-1}, \ell \in \Omega^{s-1}\}$ defined above depends critically on X_s^*. In other words, there may well exist other feasible (but not necessarily optimal) decision rules for Problem 1 that would generate a covering of $\bar{Q}_{s-1}$ different from the one given by $\{A_\ell^{s-1}, \ell \in \Omega^{s-1}\}$. However, because the chief concern is with deriving necessary, rather than sufficient, conditions for X_s^*, only $\{A_\ell^{s-1}, \ell \in \Omega^{s-1}\}$, henceforth referred to as the "optimal partition" of $\bar{Q}_{s-1}$, is considered.

It must also be emphasized that X_s^*, an optimal X_s, is not necessarily unique. That the solution of Problem 1 is not unique follows from the fact that any other decision rule X_s' that is such that $\int \cdots \int_A f_{s-1} db_1 \ldots db_{s-1} = 0$, where $A = \{(b_1, \ldots, b_{s-1}) : X_s' \neq X_s^*\}$, will satisfy the constraints of Problem 1 and have $c_s E(X_s') = c_s E(X_s^*)$. Hence X_s' will also be optimal for Problem 1; i.e., the optimal decision rule for Problem 1 is, in general, nonunique—at least on a set of measure zero. This trivial nonuniqueness can of course be avoided by adopting a convention such as making X_s^* right (or left) continuous. However, this is not the only kind of nonuniqueness that can occur. In Sec 6 a situation is illustrated in which the optimal rule fails—in a very significant way—to be unique. In fact the situation is such that two optimal rules could fail to be equal for every point $(b_1, \ldots, b_{s-1}) \in Q_{s-1}$.

In order to solve Problem 1, rewrite the ith constraint using assumption a.

$$P\left(\sum_{j=1}^{i} a_{ij} X_j + d_i b_i + \omega_i(b_1, \ldots, b_{i-1}) \geq 0\right)$$

$$= \begin{cases} P\left(b_i \geq -\sum_{j=1}^{i} \dfrac{a_{ij}}{d_i} X_j - \dfrac{1}{d_i}\omega_i\right), & \text{for } d_i > 0, \\[2ex] P\left(b_i \leq -\sum_{j=1}^{i} \dfrac{a_{ij}}{d_i} X_j - \dfrac{1}{d_i}\omega_i\right), & \text{for } d_i < 0. \end{cases}$$

Let

$$a_{ij}' = \frac{a_{ij}}{d_i} \text{ and } \omega_i' = \frac{\omega_i}{d_i}.$$

Let Γ^i be the set over which

$$b_i \geq -\sum_{j=1}^{i} a_{ij}' X_j - \omega_i'.$$

Then

$$P\left(b_i \geq -\sum_{j=1}^{i} a_{ij}' X_j - \omega_i'\right)$$

$$= \int \cdots \int_{\Gamma^i} f_n(b_1, \ldots, b_n)\, db_1 \ldots db_n,$$

by our interpretation of the P operator.

By assumption d, X_j is only a function of $b_1, \ldots, b_{j-1}$, and so the set Γ^i depends only on $b_1, \ldots, b_i$. Hence the above integration can be performed first with respect to b_j, $j = i+1, \ldots, n$ and then with respect to b_j, $j = 1, \ldots, i$. But integrating with respect to $b_j, j = i+1, \ldots, n$, we are integrating the joint frequency function of the various random variables over their entire range of possible values (i.e., over all the values that they can take on with nonzero probability). Since the value of this integral is 1,

$$\underset{\Gamma^i}{\int \cdots \int} f_n \, db_1 \ldots db_n = \underset{\Gamma^i}{\int \cdots \int} f_i \, db_1 \ldots db_i =$$

$$\underset{\bar{Q}_{i-1}}{\int \cdots \int} \bar{F}_i\left(-\sum_{j=1}^{i} a'_{ij} X_j - \omega'_i\right) f_{i-1} \, db_1 \ldots db_{i-1},$$

where $\bar{F}_i(.)$ is the conditional distribution function of b_i given $b_1, \ldots, b_{i-1}$. This last equation is obtained by holding $b_1, \ldots, b_{i-1}$ fixed and integrating over Γ^i with respect to b_i.

Similarly for $d_i < 0$

$$P\left(\sum_{j=1}^{i} a_{ij} X_j + d_i b_i + \omega_i \leq 0\right) = 1 - \underset{\bar{Q}_{i-1}}{\int \cdots \int} \bar{F}_i\left(-\sum_{j=1}^{i} a'_{ij} X_j - \omega'_i\right) f_{i-1} \, db_1 \ldots db_{i-1}.$$

$$E(c_j X_j) = c_j E(X_j) = c_j \underset{\bar{Q}_n}{\int \cdots \int} X_j f_n \, db_1 \ldots db_n = c_j \underset{\bar{Q}_{j-1}}{\int \cdots \int} X_j f_{j-1} \, db_1 \ldots db_{j-1}.$$

Therefore Problem 1 can be written in the following form:
maximize

$$\sum_{j=1}^{n} c_j \underset{\bar{Q}_{j-1}}{\int \cdots \int} X_j f_{j-1} \, db_1 \ldots db_{j-1}$$

subject to

$$\operatorname{sgn}(d_i) \underset{\bar{Q}_{i-1}}{\int \cdots \int} \bar{F}_i\left(-\sum_{j=1}^{i} a'_{ij} X_j - \omega'_i\right) f_{i-1} \, db_1 \ldots db_{i-1} \geq \alpha'_i, \; i = 1, \ldots, n$$

$$P(X_j \geq 0) \geq \beta_j, \; j = 1, \ldots, n, \tag{2}$$

where

$$\alpha'_i = \begin{cases} \alpha_i \text{ if } \operatorname{sgn}(d_i) = +1 \text{ (i.e., if } d_i > 0) \\ \alpha_i - 1 \text{ if } \operatorname{sgn}(d_i) = -1 \text{ (i.e., if } d_i < 0), \end{cases}$$

and by $\bar{Q}_0$ is meant that we perform no integration but just get $c_1 X_1$ in the objective function and $\operatorname{sgn}(d_1)\, \bar{F}_1(-a'_{11} X_1 - \omega'_1) \geq \alpha'_1$ in the constraints.

For convenience, fix on certain choices of sign for some of the constants involved in Problem 1 to carry forward the mathematical arguments. Only a simple formal exchange must be made in order to encompass the other possible choices of sign into our results. It is assumed that $d_i < 0$, $c_i \geq 0$, $a_{ii} > 0$ for

$i = 1, \ldots, n$. By defining $a''_{ij} = -a'_{ij}$, and $\omega''_i = -\omega'_i$ Problem 2 can be written as maximize

$$\sum_{j=1}^{n} c_j \int_{\bar{Q}_{j-1}} \cdots \int X_j f_{j-1} \, db_1 \ldots db_{j-1}$$

subject to

$$\int_{\bar{Q}_{i-1}} \cdots \int F_i \left(\sum_{j=1}^{i} a''_{ij} X_j + \omega''_i \right) f_{i-1} \, db_1 \ldots db_{i-1} \geq 1 - \alpha_i, \; i = 1, \ldots, n$$

$$P(X_j \geq 0) \geq \beta_j, \; j = 1, \ldots, n. \qquad (3)$$

For the sake of notational convenience the primes will be dropped from a''_{ij} and ω''_i throughout the remainder of the paper. Moreover, the region of integration will no longer be explicitly written out, but it will be implicitly understood that when we integrate using the frequency function f_{i-1} $i = 2, \ldots, n$ the region of integration, unless otherwise stated, is $\bar{Q}_{i-1}$.

Finally, assume in what follows that $|c_j E(X_j^*)| < \infty, j = 1, \ldots, n$ and that there exist decision rules X_j^*, $j = 1, \ldots, n$ that are feasible for Problem 3. Various ways of modifying a chance-constrained problem, when the constraints are inconsistent or the optimal value of the objective function is unbounded, are discussed at length in a previous paper.[7]

3. SOLUTION OF THE sth SUBPROBLEM

By the Sth subproblem the following is meant: maximize

$$c_s \int \cdots \int X_s f_{s-1} \, db_1 \ldots db_{s-1}$$

subject to

$$\int \cdots \int F_s \left(\sum_{j=1}^{s} a_{sj} X_j + \omega_s \right) f_{s-1} \, db_1 \ldots db_{s-1} \geq 1 - \alpha_s,$$

$$P(X_s \geq 0) \geq \beta_s. \qquad (4)$$

In Problem 4 it is assumed that $X_j(b_1, \ldots, b_{j-1})$, $j = 1, \ldots, s-1$ are given functions and that the optimal $X_s(b_1, \ldots, b_{s-1})$ is sought. Furthermore, it is assumed that the given functions X_j, $j = 1, \ldots, s-1$, and all feasible X_s satisfy the requirements imposed on X_j, $j = 1, \ldots, n$ in the preceding section. Thus Problem 4 is to be solved for the optimal X_s as a function of ω_s and the given decision rules $X_1, \ldots, X_{s-1}$. In Sec 4 the results of this section and an inductive argument to solve the n-period Problem 3 are used.

Let X_s^* be an optimal solution for Problem 4.

Let

$$\gamma_\ell^* = \int \cdots \int_{A_\ell^{s-1}} \bar{f}_s\left(a_{ss} X_s^* + \sum_{j=1}^{s-1} a_{sj} X_j + \omega_s\right) f_{s-1}\, db_1 \ldots db_{s-1}, \text{ for each } \ell \in \Omega^{s-1}.$$

$$I' = \{\ell : X_s^* \geq 0 \text{ in } A_\ell^{s-1}, \ell \in \Omega^{s-1}\}.$$

$$J' = \{\ell : X_s^* \leq 0 \text{ in } A_\ell^{s-1}, \ell \in \Omega^{s-1}\}.$$

Now consider the following localization of the problem:
maximize

$$c_s \int \cdots \int_{A_\ell^{s-1}} X_s f_{s-1}\, db_1 \ldots db_{s-1}$$

subject to

$$\int \cdots \int_{A_\ell^{s-1}} \bar{f}_s\left(a_{ss} X_s + \sum_{j=1}^{s-1} a_{sj} X_j + \omega_s\right) f_{s-1}\, db_1 \ldots db_{s-1} = \gamma_\ell^* . \quad (5)$$

with the additional constraint that $X_s \geq 0$ if $\ell \in I'$ and $X_s \leq 0$ if $\ell \in J'$. Clearly,

Lemma 1. The optimal continuous X_s for Problem 5 is $X_s^* | A_\ell^{s-1}$, the restriction of X_s^* to A_ℓ^{s-1}.

Proof. $X_s^* | A_\ell^{s-1}$ is feasible for Problem 5 and is continuous from our definition of A_ℓ^{s-1}. Also if $X_s^* | A_\ell^{s-1}$ was not optimal for Problem 5, then X_s^* would fail to be optimal for Problem 4, since on A_ℓ^{s-1} its contribution to the objective function of Problem 4 could be improved.

Lemma 1 shows that a necessary condition that X_s^* be an optimal decision rule for Problem 4 is that for each A_ℓ^{s-1}, $\ell \in \Omega^{s-1}$, X_s^* be the optimal continuous solution for Problem 5. Further necessary conditions on X_s^* are now obtained for Problem 5, using variational theory.

Theorem 1

A necessary condition for $X_s^* | A_\ell^{s-1}$ to be the optimal continuous X_s for Problem 5 is that for each point $(b_1, \ldots, b_{s-1})$ in A_ℓ^{s-1}, either $X_s^*(b_1, \ldots, b_{s-1}) = 0$, or $\bar{f}_s(a_{ss} X_s^* + \Sigma_{j=1}^{s-1} a_{sj} X_j + \omega_s) = - c_s \ \lambda a_{ss}$ where λ is a constant.

Proof. Let $\ell \in I'$. Then in Problem 5 we will make the change of variable $X_s = Z_s^2$. The following problem is now solved:
maximize

$$c_s \int \cdots \int_{A_\ell^{s-1}} Z_s^2 f_{s-1}\, db_1 \ldots db_{s-1} .$$

subject to

$$\int \cdots \int_{A_\ell^{s-1}} \bar{f}_s\left(a_{ss} Z_s^2 + \sum_{j=1}^{s-1} a_{sj} X_j + \omega_s\right) f_{s-1}\, db_1 \ldots db_{s-1} = \gamma_\ell^* . \quad (6)$$

for the optimal continuous function $Z_s(b_1, \ldots, b_{s-1})$.

Problem 6 is a multiple integral isoperimetric problem in the calculus of variations. In App A it is proved that the assumptions about A_ℓ^{s-1}, $\ell \in \Omega^{s-1}$

are sufficient for the derivation of the Euler equation $\partial H/\partial Z_s = 0$, where $H(b_1, \ldots, b_{s-1})$ is defined by

$$H(b_1, \ldots, b_{s-1}) = c_s Z_s^2 f_{s-1} + \lambda F_s\left(a_{ss} Z_s^2 + \sum_{j=1}^{s-1} a_{sj} X_j + c_s\right) f_{s-1}$$

and $\lambda \neq 0$ is a constant.

Since the Euler equation provides a necessary condition for Z_s^* to be optimal for Problem 6, $\partial H(Z_s^*)/\partial Z_s = 0$ implies

$$2Z_s^* f_{s-1}\left[c_s + \lambda a_{ss} F_s'\left(a_{ss} Z_s^{*2} + \sum_{j=1}^{s-1} a_{sj} X_j + c_s\right)\right] = 0 , \tag{7}$$

and Eq 7 must hold for all points in A_ℓ^{s-1}.

Since any point $(b_1, \ldots, b_{s-1})$ for which $f_{s-1} = 0$ contributes nothing to the objective function or the constraint of Problem 6, only points for which $f_{s-1} > 0$ will be considered. Hence we conclude from Eq 7 that at such points

$$\begin{cases} \text{either} & Z_s^* = 0 , \\ \text{or} & F_s'\left(a_{ss} Z_s^{*2} + \sum_{j=1}^{s-1} a_{sj} X_j + c_s\right) = \dfrac{-c_s}{\lambda a_{ss}} = \text{a constant}, \end{cases}$$

i.e.,

$$\begin{cases} \text{either} & X_s^* = 0 , \\ \text{or} & F_s'\left(a_{ss} X_s^* + \sum_{j=1}^{s-1} a_{sj} X_j + c_s\right) = \dfrac{-c_s}{\lambda a_{ss}} . \end{cases} \tag{8b}$$

Since either Eq 8a or Eq 8b must hold for every point in A_ℓ^{s-1}, the theorem is proved for $\ell \in I'$. Since a similar result can be obtained if $\ell \in J'$, the theorem is proved for all $\ell \in L^{s-1}$.

The set of rectangles $\{A_\ell^{s-1}, \ell \in L^{s-1}\}$ is now redefined to be such that, in addition to having properties i–v of Sec 2, they also have the property that

(vi) either X_s^* is identically zero in A_ℓ^{s-1} or else

$$X_s^* = \sum_{j=1}^{s-1} \frac{a_{sj}}{a_{ss}} X_j - \frac{c_s}{a_{ss}} + \frac{1}{a_{ss}} f_s^{-1}(\Gamma_\ell^{s*}) \text{ in } A_\ell^{s-1} ,$$

where Γ_ℓ^{s*} is a constant such that there exists a solution $Z^*(b_1, \ldots, b_{s-1})$ of the equation $f_s(Z^*) = \Gamma_\ell^{s*}$ for each point $(b_1, \ldots, b_{s-1}) \epsilon A_\ell^{s-1}$, and $Z^*(b_1, \ldots, b_{s-1}) = f_s^{-1}(\Gamma_\ell^{s*})$ is defined for each point $(b_1, \ldots, b_{s-1})$ by $Z^*(b_1, \ldots, b_{s-1}) = \max \{Z : f_s(Z) = \Gamma_\ell^{s*}\}$. That such a set of rectangles exists follows from the fact that in each A_ℓ^{s-1} defined in Sec 2 X_s^* is continuous; hence, by Theorem 1, each A_ℓ^{s-1} can be partitioned into a set of rectangles, say $A_\ell^{s-1} = \bigcup_{r \epsilon R} A_{\ell,r}^{s-1}$, such that in each $A_{\ell,r}^{s-1}$ properties i to v and property vi hold. The subscript r can then simply be dropped to make notation easier. Thus Corollary 1 is obtained.

Corollary 1. A necessary condition that X_s^* be the optimal X_s for Problem 4 is that there exists a set of rectangles $\{A_\ell^{s-1}, \ell \epsilon L^{s-1}\}$ with properties i–vi.

Let $I = \{\ell : X_s^* \geq 0$ with strict inequality for some points in $A_\ell^{s-1}\}$.
Let $J = \{\ell : X_s^* \leq 0$ with strict inequality for some points in $A_\ell^{s-1}\}$.
Let $K = \{\ell : X_s^*$ is identically zero in $A_\ell^{s-1}\}$.

Consider the problem
maximize

$$\sum_{\ell \in I,J} \frac{c_s}{a_{ss}} \int \cdots \int_{A_\ell^{s-1}} \bar{f}_s^{-1}(T_\ell^s)\, f_{s-1}\, db_1 \ldots db_{s-1}$$

$$- \sum_{\ell \in I,J} c_s \int \cdots \int_{A_\ell^{s-1}} \left[\sum_{j=1}^{s-1} \frac{a_{sj}}{a_{ss}} X_j + \frac{\omega_s}{a_{ss}} \right] f_{s-1}\, db_1 \ldots db_{s-1} \tag{9}$$

subject to

$$\sum_{\ell \in I,J} \int \cdots \int_{A_\ell^{s-1}} \bar{F}_s\left(\bar{f}_s^{-1}(T_\ell^s)\right) f_{s-1}\, db_1 \ldots db_{s-1} = 1 - \alpha_s - X, \tag{9a}$$

$$\bar{f}_s^{-1}(T_\ell^s) \geq \sum_{j=1}^{s-1} a_{sj} X_j + \omega_s, \text{ all } (b_1, \ldots, b_{s-1}) \in A_\ell^{s-1} \text{ and } \ell \in I, \tag{9b}$$

$$\bar{f}_s^{-1}(T_\ell^s) \leq \sum_{j=1}^{s-1} a_{sj} X_j + \omega_s, \text{ all } (b_1, \ldots, b_{s-1}) \in A_\ell^{s-1} \text{ and } \ell \in J, \tag{9c}$$

$$V_\ell^s \leq T_\ell^s \leq W_\ell^s, \ \ell \in I, \ \ell \in J, \tag{9d}$$

where V_ℓ^s, W_ℓ^s are constants and

$$X = \sum_{\ell \in K} \int \cdots \int_{A_\ell^{s-1}} \bar{F}_s\left(\sum_{j=1}^{s-1} a_{sj} X_j + \omega_s \right) f_{s-1}\, db_1 \ldots db_{s-1}.$$

Problem 9 was obtained by using the expression for X_s^* given by Corollary 1. Problem 9a corresponds to the first constraint of Problem 4; Problems 9b and 9c express the fact that $X_s^* \geq 0$ in A_ℓ^{s-1} for $\ell \in I$, and $X_s^* \leq 0$ in A_ℓ^{s-1} for $\ell \in J$; Problem 9d gives upper and lower bounds on T_ℓ^s in A_ℓ^{s-1}, which assure us that $\bar{f}_s^{-1}(T_\ell^s)$ is defined in A_ℓ^{s-1}, i.e., that there exists a solution of $\bar{f}_s(Z) = T_\ell^s$ for each point in A_ℓ^{s-1} for $\ell \in I$, $\ell \in J$.

Now find T_ℓ^s, $\ell \in I$, $\ell \in J$, as a function of A_ℓ^{s-1} and thus reduce Problem 4 to determining the optimal partition $\{A_\ell^{s-1}, \ell \in \Omega^{s-1}\}$.

Clearly,

Lemma 2. T_ℓ^{s*}, $\ell \in I$, $\ell \in J$, are the optimal T_ℓ^s for Problem 9.

Proof. Proceeding exactly as in the proof of Lemma 1 easily proves that if T_ℓ^{s*} are not optimal for Problem 9, then a contradiction exists, since this implies that X_s^* is not optimal for Problem 4.

Using the fact that we are trying to find T_ℓ^{s*} as a function of A_ℓ^{s-1} and that $\bar{f}_s^{-1}(\cdot)$, ω_s, and X_j, $j = 1, \ldots, s-1$ are known functions of $b_1, \ldots, b_{s-1}$, permits replacement of Problem 9b and Problem 9c by conditions that give upper and lower bounds on T_ℓ^s. Combining these new bounds with Problem 9d and dropping

the second term in the objective function of Problem 9, since it is independent of T_ℓ^s, permits writing Problem 9 as
maximize

$$\sum_{\ell \in I, J} \int \cdots \int_{V_\ell^{s-1}} F_s\left[\bar{f}_s^{-1}(T_\ell^s)\right] f_{s-1}\, db_1 \ldots db_{s-1} \geq 1 - \alpha_s - \lambda .$$

subject to

$$L_\ell^s \leq T_\ell^s \leq U_\ell^s,\ \ell \in I,\ \ell \in J, \tag{10}$$

where L_ℓ^s U_ℓ^s are constants.

Employing a simple Lagrange multiplier argument makes it easy to establish that a necessary condition that T_ℓ^{s*} be optimal for Problem 10 is that T_ℓ^{s*} take on one of the three values L_ℓ^s, U_ℓ^s, or T^{s*}, where T^{s*} is a constant that does not depend on ℓ and so $L_\ell^s \leq T^{s*} \leq U_\ell^s$ for all $\ell \in I, J$. Moreover, if T_ℓ^{s*} is equal to L_ℓ^s or U_ℓ^s, then it is also equal to V_ℓ^s or W_ℓ^s. In other words, T_ℓ^{s*} cannot equal the bounds on T_ℓ^s obtained from Problem 9b or Problem 9c unless these bounds are the same as those given by Problem 9d.

Thus Theorem 2 is proved.

<u>Theorem 2</u>

There exists some constant, say T^{s*}, such that T_ℓ^{s*} takes on one of the three values V_ℓ^s, W_ℓ^s, or T^{s*}, and $V_\ell^s \leq T^{s*} \leq W_\ell^s$ for $\ell \in I$, $\ell \in J$.

4. THE n-PERIOD PROBLEM

<u>Theorem 3</u>

A necessary condition for X_s^*, $j = 2, \ldots, n$ to be optimal decision rules for Problem 3 is that for each j a set of rectangles $\{A_\ell^{s-1}, \ell \in \Omega^{s-1}\}$ exists with properties i–vi as defined above, with X_j replaced by X_j^* in iv and vi.

Proof. The theorem is proved by induction on t where $s = n + 1 - t$. Begin by proving the theorem for $t = 1$, i.e., that it is true for X_n^*. Then assume for induction that the theorem is true for $t = k$, i.e., for $X_{n-k+1}, \ldots, X_n^*$, and then prove it is true for $t = k + 1$, i.e., for X_{n-k}^*.

Let $t = 1$.

Let $X_1, \ldots, X_{n-1}$ be feasible decision rules for the first $n - 1$ periods. Then the problem of determining X_n^* is equivalent to solving Problem 4 with $s = n$. Using Corollary 1, it can be seen that Theorem 3 is true for $t = 1$.

Assume for induction that the theorem is true when $t = k$, i.e., that it holds for $X_{n-k+1}^*, \ldots, X_n^*$. Now prove the theorem is true for $t = k + 1$, i.e., for X_{n-k}^*.

Using the induction hypothesis, the expressions for $X_{n-k+1}^*, \ldots, X_n^*$ can be put into the objective function of Problem 3 to get

$$\sum_{j=n-k+1}^{n} c_j E(X_j^*) = \sum_{j=n-k+1}^{n} c_j \left\{ \sum_\ell \int \cdots \int_{V_\ell^{j-1}} \left[- \sum_{i=1}^{j-1} \frac{a_{ji}}{a_{jj}} X_i^* - \frac{c_j}{a_{jj}} \cdot \frac{\bar{f}_j^{-1}(T_\ell^{j*})}{a_{jj}} \right] f_{j-1} \prod_{i=1}^{j-1} db_i \right\} \tag{11}$$

in which only those $\ell \in \Omega^{j-1}$ for which X_{j-1}^* is not identically zero are summed over. By integrating Eq 11 with respect to b_{j-1}, $j = n-k+1, \ldots, n$, the resulting integral, with respect to $b_1, \ldots, b_{n-k-1}$, is such that the integrand is a piecewise linear function of X_i^*, $i = 1, \ldots, n-k$. To see this the right-hand side of Eq 11 is written as

$$\sum_{i=1}^{n-k} \sum_{j=n-k+1}^{n} \sum_{\ell} \int \cdots \int_{A_\ell^{j-1}} X_i^* \left\{ -c_j k_{ji} \right\} f_{j-1}\, db_1, \ldots, db_{j-1}$$

$$+ \sum_{j=n-k+1}^{n} c_j \left\{ \sum_{\ell} \int \cdots \int_{A_\ell^{j-1}} \left[-\frac{\omega_j}{a_{jj}} + \frac{\bar{f}_j^{-1}(T_\ell^{j*})}{a_{jj}} \right] f_{j-1}\, db_1, \ldots, db_{j-1} \right\}, \quad (12)$$

where k_{ji} is a constant that depends on the various a_{ji}. However, it is known that X_i^* is a function of only $b_1, \ldots, b_{i-1}$, so that for each i we can perform the integration in the first term of expression 12 by integrating first with respect to $b_i, \ldots, b_{j-1}$, and then with respect to $b_1, \ldots, b_{i-1}$. Integrating with respect to $b_i, \ldots, b_{j-1}$, yields an expression for the first term of expression 12 that is of the form

$$\sum_{i=1}^{n-k} \sum_{\ell} \int \cdots \int_{B_\ell^{i-1}} c_\ell^i X_i^*\, db_1, \ldots, db_{i-1}, \quad (12a)$$

where the sets B_ℓ^{i-1} result from integrating over the various A_ℓ^{j-1} with respect to $b_i, \ldots, b_{j-1}$, and the coefficients c_ℓ^i depend on ℓ due to the effect of integrating f_{j-1}, with respect to $b_i, \ldots, b_{j-1}$, over the set A_ℓ^{j-1}.

Now suppose that $X_1, \ldots, X_{n-k-1}$ are decision rules that are feasible for the first $n-k-1$ periods. Then the problem of finding X_{n-k}^* is of the form maximize

$$\sum_{\ell \in \Omega^{n-k-1}} c_\ell^{n-k} \int \cdots \int_{A^{n-k-1}} X_{n-k} f_{n-k-1}\, db_1 \ldots db_{n-k-1}$$

subject to

$$\sum_{\ell \in \Omega^{n-k-1}} \int \cdots \int_{A_\ell^{n-k-1}} \left[F_{n-k}\left(a_{n-k,n-k} X_{n-k} + \sum_{i=1}^{n-k-1} a_{n-k,i} X_i + \omega_{n-k} \right) \right] f_{n-k-1} \prod_{i=1}^{n-k-1} db_i \geq 1 - \alpha_{n-k}$$

$$P(X_{n-k} \geq 0)\, \beta_{n-k}, \quad (13)$$

where the c_ℓ^{n-k} depend on ℓ, as explained above. However, c_ℓ^{n-k} is constant for each ℓ and hence Problem 13 can be written as a series of problems (one for each $\ell \in \Omega^{n-k-1}$), each of which is equivalent to Problem 5 with $s = n-k$ and $c_s = c_\ell^{n-k}$. Hence the results of Corollary 1 can be applied again, and hence Theorem 3 has been proved for $t = k+1$.

Therefore, the theorem is proved by induction.

Corollary 2. X_j^*, $j = 1, \ldots, n$ is a piecewise linear function of ω_i, $i = 1, \ldots, j$ and $\bar{f}^{-1}(T_\ell^{i*})$, $\ell \in \Omega^{i-1}$, $i = 1, \ldots, j$.

Proof. By Theorem 3, X_j^* is either zero or a linear function of X_k^*, $k = 1, \ldots, j-1$, ω_j, and $\bar{f}_j^{-1}(T_\ell^{j*})$; hence it is a piecewise linear function of ω_j, and $\bar{f}_j^{-1}(T_\ell^{j*})$, and X_k^*, $k = 1, \ldots, j-1$. Since this is also true for X_k^*, $k = 1, \ldots, j-1$, the corollary is proved.

Unfortunately the fact that c_ℓ^{n-k} in Problem 13 depends on ℓ makes it impossible to extend the results of Theorem 2 to the general n-stage problem. For in this case the Lagrangian solution of Problem 10 will yield a T^{s*} that depends on ℓ for the same reason that the coefficient c_ℓ^{n-k} in the objective function of Problem 13 depends on ℓ.

However, the following theorem can be proved. An alternative and somewhat simpler proof is given in App B.

Theorem 4

If none of the constraints $P(X_j \geq 0) \geq \beta_j, j = 1, \ldots, n$ in Problem 3 are tight, then for each j, $j = 1, \ldots, n$, T_ℓ^{j*} defined in Theorem 3 can take on only one of the three values V_ℓ^j, or W_ℓ^j, or T^{j*}, where V_ℓ^j, W_ℓ^j are defined as in Theorem 2 and $V_\ell^j > T^{j*} > W_\ell^j$ for all $\ell \epsilon \Omega^{j-1}$.

Proof. Suppose the constraints $P\{X_j \geq 0\} \geq \beta_j$, $j = 1, \ldots, n$ are not binding in Problem 3. Then the sign of X_j^* in any set A_ℓ^{j-1}, $\ell \epsilon \Omega^{j-1}$ need no longer be of concern. In particular, in Problem 5 the additional constraint that $X_s \geq 0$ if $\ell \epsilon I'$ and $X_s < 0$ if $\ell \epsilon J'$ is not needed.

Thus in the proof of Theorem 1 the change of variable $X_s = Z_s^2$ need not be made since the sign of X_s in A_ℓ^{s-1} can be allowed to change. Again, using variational theory gives in place of Eq 7 that $\partial H / \partial X_s = 0$ implies

$$c_s + \lambda a_{ss} \bar{f}_s \left(a_{ss} X_s^* + \sum_{j=1}^{s-1} a_{sj} X_j + \alpha_s \right) = 0 . \tag{7a}$$

Thus only Eq 8b can hold and hence

$$X_s^* = -\sum_{j=1}^{s-1} \frac{a_{sj}}{a_{ss}} X_j - \frac{\alpha_s}{a_{ss}} + \frac{1}{a_{ss}} \bar{f}_s^{-1} (T_\ell^{s*}) \tag{14}$$

in each A_ℓ^{s-1}.

Theorem 2 can then be proved as above.

Again Theorem 3 is seen to hold for $t = 1$ and, assuming it is true for $t = k$, the effect of $X_{n-k+1}^*, \ldots, X_n^*$ on the objective function of Problem 13 is to make c_ℓ^{n-k} independent of ℓ, (i.e., $c_\ell^{n-k} = c^{n-k}$ for all $\ell \epsilon \Omega^{n-k}$, where c^{n-k} is a constant). This is true because Eq 14 implies that X_j^*, $j = n-k+1, \ldots, n$ is strictly linear in X_{n-k}, not piecewise linear as it was in the previous case. Hence in Eq 11, summing is over all $\ell \epsilon \Omega^{j-1}$, and hence the first term of expression 12 can be written in the form

$$\sum_{i=1}^{n-k} \sum_{j=n-k+1}^{n} \int \cdots \int_{Q_{j-1}} X_i^* \{ -c_j k_{ji} \} f_{j-1} \, db_1 \cdots db_{j-1} ,$$

where the sum of the integrals over all A_ℓ^{j-1} has been dropped and replaced by an integration over Q_{j-1}, since it is known that $Q_{j-1} = \bigcup_{\ell \epsilon \Omega^{j-1}} A_\ell^{j-1}$. But in this case when integration is performed with respect to $b_1, \ldots, b_{j-1}$ the resulting value of the integral is 1, since the integration is performed over all possible values of these random variables, not just some of the values as in the proof of Theorem 3. Hence, in place of expression 12a, there is obtained

$$\sum_{i=1}^{n-k} \int \cdots \int_{Q_{i-1}} c^i X_i^* \, db_1 \cdots db_{i-1} .$$

where c_i is a constant.

This means that the problem that must be solved to determine X^*_{n-k} is the same as Problem 4 with $s = n-k$ and $c_s = c^{n-k}$. Thus Corollary 1 can be used to find X^*_{n-k}, and hence Theorem 4 is shown to be true for $t = k+1$.

Thus the theorem is proved by induction.

5. AN EXTENSION OF THE RESULTS

In this section an extension of Theorem 3 is established. Suppose that c_i, $i = 1, \ldots, n$ are continuous random variables. If $f_i \equiv f_i(b_1, \ldots, b_i, c_1, \ldots, c_i)$ = the joint frequency function of the random variables b_j, c_j, $j = 1, \ldots, i$, if it is assumed that f_n is a known frequency function, and if $X_i \equiv X_i(b_1, \ldots, b_{i-1}, c_1, \ldots, c_{i-1})$; then Problem 2 becomes

maximize

$$\sum_{j=1}^{n} \underset{\bar{Q}_{j-1}}{\int \cdots \int} \bar{c}_j X_j f_{j-1} \prod_{k=1}^{j-1} d(b_k)\, d(c_{jk})$$

subject to

$$\operatorname{sgn}(d_i) \underset{\bar{Q}_{i-1}}{\int \cdots \int} \bar{F}_i \left[- \sum_{j=1}^{i} \frac{a_{ij}}{d_i} X_j - \frac{\omega_i}{d_i} \right] f_{i-1} \prod_{k=1}^{i-1} d(b_k)\, d(c_k) \geq \alpha_i \quad i = 1, \ldots, n,$$

$$P(X_j \geq 0) \geq \beta_j, \; j = 1, \ldots, n,$$

where $\bar{Q}_{i-1}$ is the closure of the set in $2(i-1)$-dimensional Euclidean space where $f_{i-1} > 0$, and $\bar{c}_j$ is the conditional expectation of c_j given b_k, c_k, $k = 1, \ldots, j-1$.

Then it can be established that Lemma 1 continues to hold, only now $\{A^{s-1}_\ell, \ell \epsilon \mathfrak{L}^{s-1}\}$ is a set of $2(s-1)$-dimensional rectangles. Theorem 1 is also true in this case, except that $-\bar{c}_s/a_{ss}$ is no longer a constant but rather a function of the conditional random variables involved in $\bar{c}_s$.

Thus Problem 9 is no longer a problem in determining a constant T^s_ℓ but rather one of finding a function $\overset{s}{T_\ell}$, and hence the Lagrange multiplier technique used to establish Theorem 2 will not work.

However, Theorem 3 can be proved just as was done above by replacing c^i_ℓ by $\overset{-1}{c_\ell}$. Thus the following result has been established:

<u>Theorem 5</u>

If in Problem 3 it is assumed that c_j, b_i, $i, j = 1, \ldots, n$ are continuous random variables, then X^*_j, $j = 1, \ldots, n$ is a piecewise linear function of ω_i and $\bar{f}_i^{-1}(T^{i*}_\ell)$, $i = 1, \ldots, j$, where T^{i*}_ℓ is a function of $b_1, \ldots, b_{i-1}, c_1, \ldots, c_{i-1}$.

This is the specialization of Theorem 2 in our previous paper[1] to the triangular case.

6. INDEPENDENT RANDOM VARIABLES

Return again to the problem considered in Sec 2, in which the c_j, $j = 1, \ldots, n$ are constants. Also introduce the additional assumption that the random variables b_i, $i = 1, \ldots, n$ are mutually independent. Let $\tilde{f}_i(.)$, and $\tilde{F}_i(.)$, $i = 1, \ldots, n$ represent the frequency function and the distribution function respectively of the continuous random variable b_i, $i = 1, \ldots, n$.

In this case several extensions of the previous results are immediately available. First, owing to the assumption about the independence of the b_i, $\bar{f}_s(.) = \tilde{f}_s(.)$; hence $\bar{f}_s^{-1}(T_\ell^{s*})$ defined in Corollary 1 equals $\tilde{f}_s^{-1}(T_\ell^{s*})$, which is a constant, i.e., not a function of any b_i. Moreover, using the definition of γ_ℓ^* and the fact that $\bar{F}_s[\bar{f}_s^{-1}(T_\ell^{s*})] = \tilde{F}_s[\tilde{f}_s^{-1}(T_\ell^{s*})]$ is independent of $b_1, \ldots, b_{s-1}$, gives $\tilde{F}_s[\tilde{f}^{-1}(T_\ell^{s*})] F_{s-1}(A_\ell^{s-1}) = \gamma_\ell^*$ for $\ell \epsilon I$ and $\ell \epsilon J$. This implies that

$$\tilde{f}_s^{-1}(T_\ell^{s*}) = \tilde{F}_s^{-1}\left(\frac{\gamma_\ell^*}{F_{s-1}(A_\ell^{s-1})}\right) \quad \tilde{F}_s^{-1}(D_\ell^{s*}),$$

where D_ℓ^{s*} is some constant in $[0,1]$.

Using this definition of $\tilde{F}_s^{-1}(D_\ell^{s*})$ permits replacement of $\bar{f}_s^{-1}(T_\ell^{s*})$ by $\tilde{F}_s^{-1}(D_\ell^{s*})$ in property vi of $\{A_\ell^{s-1}, \ell \epsilon \Omega^{s-1}\}$, which was defined following Theorem 1.

Thus the equivalent of Problem 9 is

maximize

$$\left\{ \sum_{\ell \epsilon I,J} \frac{c_s}{a_{ss}} \tilde{F}^{-1}(D_\ell^s) F_{s-1}(A_\ell^{s-1}) \right.$$

$$\left. - \sum_{\ell \epsilon I,J} c_s \int \cdots \int_{A_\ell^{s-1}} \left[\sum_{j=1}^{s-1} \frac{a_{sj}}{a_{ss}} X_j + \frac{\omega_s}{a_{ss}} \right] f_{s-1}\, db_1 \ldots db_{s-1} \right\} \tag{15}$$

subject to

$$\sum_{\ell \epsilon I,J} D_\ell^s F_{s-1}(A_\ell^{s-1}) \leq 1 - \alpha_s - X, \tag{15a}$$

$$D_\ell^s \geq \tilde{F}_s\left(\sum_{j=1}^{s-1} a_{sj} X_j + \omega_s\right), \text{ all } (b_1, \ldots, b_{s-1}) \epsilon A_\ell^{s-1} \text{ and } \ell \epsilon I, \tag{15b}$$

$$D_\ell^s \leq \tilde{F}_s\left(\sum_{j=1}^{s-1} a_{sj} X_j + \omega_s\right), \text{ all } (b_1, \ldots, b_{s-1}) \epsilon A_\ell^{s-1} \text{ and } \ell \epsilon J, \tag{15c}$$

$$0 \leq D_\ell^s \leq 1, \text{ all } \ell. \tag{15d}$$

It is clear that D_ℓ^{s*} are the optimal D_ℓ^s for Problem 15.

In solving Problem 15, 15b and 15c constraints can be replaced by constraints of the form

$$D_\ell^s \geq \tilde{F}_s(k_\ell^{s-1}), \text{ for } \ell \epsilon I, \tag{16}$$

and

$$D_\ell^s \leq \tilde{F}_s(k_\ell^{s-1}), \text{ for } \ell \epsilon J, \tag{17}$$

where k_ℓ^{s-1} is a constant that depends on A_ℓ^{s-1}.

Since these constraints give bounds on D_ℓ^s, Problem 15 can be written in a form similar to Problem 10. This problem is then solved with the result that D_ℓ^{s*} can take on only one of the three values, 0, 1, or D^{s*}, where $0 < D^{s*} < 1$. This is the analog of Theorem 2.

If $\tilde{f}_j^{-1}(T_\ell^{j*})$ is replaced by $\tilde{F}_j^{-1}(D_\ell^{j*})$, the results of Theorems 3 and 4 can be obtained just as they were in Sec 4, only now, in Theorem 4, D_ℓ^{j*} can take on only the values 0, 1, or D^{j*} for $j = 1, \ldots, n$.

Let $I_1 = \{\ell : D_\ell^{j*} = D^{j*} \text{ for } \ell \epsilon \Omega^{j-1}\}$.

Let $I_2 = \{\ell : D_\ell^{j*} = 0 \text{ for } \ell \epsilon \Omega^{j-1}\}$.

Let $I_3 = \{\ell : D_\ell^{j*} = 1 \text{ for } \ell \epsilon \Omega^{j-1}\}$.

Then when Theorem 4 is applicable (i.e., when the constraints $P(X_j \geq 0) \geq \beta_j$, $j = 1, \ldots, n$, are not binding),

$$E(X_j^*) = \sum_{\ell \epsilon I_1} \int \cdots \int_{A_\ell^{j-1}} \left(- \sum_{k=1}^{j-1} \frac{a_{jk}}{a_{jj}} X_k^* - \frac{\omega_j}{a_{jj}} + \tilde{F}_j^{-1}(D^{j*}) \right) f_{j-1}\, db_1 \ldots db_{j-1}$$

$$+ \sum_{\ell \epsilon I_2} \int \cdots \int_{A_\ell^{j-1}} \left(- \sum_{k=1}^{j-1} \frac{a_{jk}}{a_{jj}} X_k^* - \frac{\omega_j}{a_{jj}} + \tilde{F}_j^{-1}(0) \right) f_{j-1}\, db_1 \ldots db_{j-1}$$

$$+ \sum_{\ell \epsilon I_3} \int \cdots \int_{A_\ell^{j-1}} \left(- \sum_{k=1}^{j-1} \frac{a_{jk}}{a_{jj}} X_k^* - \frac{\omega_j}{a_{jj}} + \tilde{F}_j^{-1}(1) \right) f_{j-1}\, db_1 \ldots db_{j-1}$$

$$= \sum_{\ell \epsilon I_1, I_2, I_3} \int \cdots \int_{A_\ell^{j-1}} \left(- \sum_{k=1}^{j-1} \frac{a_{jk}}{a_{jj}} X_k - \frac{\omega_j}{a_{jj}} \right) f_{j-1}\, db_1 \ldots db_{j-1}$$

$$+ \tilde{F}_j^{-1}(D^{j*}) \sum_{\ell \epsilon I_1} F_{j-1}(A_\ell^{j-1}) + \tilde{F}_j^{-1}(0) \sum_{\ell \epsilon I_2} F_{j-1}(A_\ell^{j-1}) + \tilde{F}_j^{-1}(1) \sum_{\ell \epsilon I_3} F_{j-1}(A_\ell^{j-1}) .$$

However, it is known that $\bar{Q}_{j-1} = \bigcup_{\ell \epsilon \Omega^{j-1}} A_\ell^{j-1}$ and that I_1, I_2, I_3 partition the set of indexes $\ell \epsilon \Omega^{j-1}$. Thus the first term in this expression does not depend on the choices of A^{j-1}. In other words this first term is known when X_k^*, $k = 1, \ldots, j-1$ and is independent of the choice of $\{A_\ell^{j-1}, \ell \epsilon \Omega^{j-1}\}$ and D^{j*}. Moreover, using the expression for X_j^*,

$$\int \cdots \int F_j\left(a_{jj} X_j^* + \sum_{k=1}^{j-1} a_{jk} X_k^* + \omega_j\right) f_{j-1}\, db_1 \ldots db_{j-1}$$

$$= D^j \sum_{\ell \epsilon I_1} F_{j-1}(A_\ell^{j-1}) + \sum_{\ell \epsilon I_3} F_{j-1}(A_\ell^{j-1}) .$$

Hence, when the constraints $P(X_j \geq 0) \geq \beta_j$, $j = 1, \ldots, n$ are not binding, to find the $\{A_\ell^{j-1}, \ell \epsilon \Omega^{j-1}\}$ and D^{j*} only this problem needs to be solved: maximize

$$\tilde{F}_j^{-1}(D^j) \sum_{\ell \epsilon I_1} F_{j-1}(A_\ell^{j-1}) + \tilde{F}_j^{-1}(0) \sum_{\ell \epsilon I_2} F_{j-1}(A_\ell^{j-1}) + \tilde{F}_j^{-1}(1) \sum_{\ell \epsilon I_3} F_{j-1}(A_\ell^{j-1})$$

subject to

$$D^j \sum_{\ell \in I_1} F_{j-1}(X_\ell^{j-1}) + \sum_{\ell \in I_3} F_j(X_\ell^{j-1}) \ge 1 - \alpha_j ,$$

$$0 \le D^j \le 1 \text{ and } \sum_{\ell \in I_1, I_2, I_3} F_{j-1}(X_\ell^{j-1}) = 1 . \tag{18}$$

Now see that by defining

$$G_{1j} = \sum_{\ell \in I_1} F_{j-1}(X_\ell^{j-1}) ,$$

$$G_{2j} = \sum_{\ell \in I_2} F_{j-1}(X_\ell^{j-1}) ,$$

$$G_{3j} = \sum_{\ell \in I_3} F_{j-1}(X_\ell^{j-1}) ,$$

and assuming D^{j*} is known, Problem 18 can be written as
maximize

$$F_j^{-1}(D^{j*})G_{1j} + \tilde{F}_j^{-1}(0)G_{2j} + \tilde{F}_j^{-1}(1)G_{3j}$$

subject to

$$D^{j*}G_{1j} + G_{3j} \ge 1 - \alpha_j ,$$

$$G_{1j} + G_{2j} + G_{3j} = 1 , \tag{19}$$

$$G_{ij} \ge 0 , \quad i = 1, 2, 3 .$$

Problem 19 is a linear programming problem in G_{ij}, $i = 1, 2, 3$. Since there are three variables and only two constraints, it is known from the theory of linear programming that at the optimal solution at least one of the $G_{ij} = 0$, $i = 1, 2, 3$. Noting that $\tilde{F}_j^{-1}(0) \ge \tilde{F}_j^{-1}(D^{j*}) \ge \tilde{F}_j^{-1}(1)$ as $0 \le D^{j*} \le 1$ and that $\tilde{F}_j^{-1}$ is a nondecreasing function, it can be seen that at the optimum $G_{2j} = 0$.

Moreover, the first constraint of Problem 19 must be satisfied as an equality at the optimum; otherwise D^{j*} could be increased, thus increasing the value of the objective function and so contradicting the assumption of optimality of D^{j*}. Therefore it may be found from the constraints of Problem 19 that

$$G_{1j}^* = \frac{\alpha_j}{1 - D^{j*}} \text{ and } G_{3j}^* = 1 - \frac{\alpha_j}{1 - D^{j*}} \tag{20}$$

are the optimal values of G_{1j} and G_{3j} respectively. These give expressions for the optimal G_{ij}, $i = 1, 2, 3$ in terms of D^{j*}.

It remains to determine D^{j*} by solving
maximize

$$F_j^{-1}(D^j)\frac{\alpha_j}{1 - D^j} + \tilde{F}_j^{-1}(1)\left[1 - \frac{\alpha_j}{1 - D^j}\right]$$

subject to

$$0 \le D^j \le 1 - \alpha_j . \tag{21}$$

By solving this nonlinear programming Problem 21, D^{j*} is obtained, and by using Eq 20, G_{1j}^* and G_{3j}^* are obtained. Thus X_j^* has been obtained explicitly for the case where Theorem 4 is applicable (i.e., where the constraints $P(X_j \geq 0) \geq \beta_j$, $j = 1, \ldots, n$ are not tight), and the random variables are independent.

Moreover, this entire development did not depend on X_k^*, $k = 1, \ldots, j-1$ since Problem 18 does not explicitly involve the decision rules of the preceding periods. Thus the results on X_j^* are valid for all j, $j = 1, \ldots, n$, and hence Theorem 6 has been proved.

<u>Theorem 6</u>

If the constraints $P\{X_j \geq 0\} \geq \beta_j$, $j = 1, \ldots, n$ are not tight, and if the random variables b_i, $i = 1, \ldots, n$ are mutually independent, then the optimal decision rules for Problem 3 are given by

$$X_j^* = -\sum_{k=1}^{j-1} \frac{a_{jk}}{a_{jj}} X_k^* + \frac{1}{a_{jj}} F_j^{-1}(D_\ell^{j*}) \text{ on } X_\ell^{j-1}\,,$$

where D_ℓ^{j*} is either 1 or D^{j*}, and $\{X_\ell^{j-1}, \ell \in Y^{j-1}\}$ are any sets for which $\bar{Q}_{j-1} = \bigcup_{\ell \in Y^{j-1}} X_\ell^{j-1}$ and that satisfy $\Sigma_{I_1} F_j(X_\ell^{j-1}) = G_{1j}^*$ and $\Sigma_{I_3} F_j(X_\ell^{j-1}) = G_{3j}^*$. Moreover, D^{j*} is found by solving Problem 21, and G_{1j}^*, G_{3j}^* are obtained from Eq 20.

Thus it has been shown that Problem 3 can be reduced to a problem of solving n rather simple nonlinear programming problems of the form of Problem 21. In particular, if each random variable b_j, $j = 1, \ldots, n$ has the same distribution, then Problem 21 needs to be solved only once to obtain D^{j*} as a function of α_j. This will then give D^{j*}, $j = 1, \ldots, n$ by putting the corresponding α_j, $j = 1, \ldots, n$ into the expression for $D^{j*}(\alpha_j)$.

It is important to note in this development that, as implied by Theorem 6, $\{X_\ell^{j-1}, \ell \in Y^{j-1}\}$ is not necessarily unique. Indeed, only the optimal covering of $\bar{Q}_{j-1}$ need be selected, subject to the restriction that G_{3j}^* and G_{1j}^* have their required values. Thus the question arises as to when this optimal covering will be unique. From Eq 20 it can be seen that this will happen only if $D^{j*} = 1 - \alpha_j$, in which case $G_{1j}^* = 1$, $G_{3j}^* = 0$, and hence the optimal decision rule is

$$X_j^* = -\sum_{k=1}^{j-1} \frac{a_{jk}}{a_{jj}} X_k^* + \frac{1}{a_{jj}} F_j^{-1}(1 - \alpha_j)$$

for all $(b_1, \ldots, b_{j-1}) \in Q_{j-1}$.

This development also shows that if $G_{3j}^* \neq 0$, so that the optimal covering of $\bar{Q}_{j-1}$ is not unique, and if $G_{1j}^* = \frac{1}{2} = G_{3j}^*$, then, in general, two optimal decision rules for X_j that do not coincide anywhere will exist in Problem 3.

Another result that is worth noting is

<u>Theorem 7</u>

If $b_1, \ldots, b_n$ are independent random variables, then a necessary condition that $D_\ell^{j*} = 1$ for some $\ell \in Y^{j-1}$ is that

$$F_j^{-1}(1) [illegible]$$

and

$$\frac{d}{db_j}\left[f_s\left(F_s^{-1}(0)\right)\right] \geq 0. \tag{22}$$

This theorem can easily be proved by using Lagrange multipliers to solve Problem 15. This result is true for the case $n = 2$ even when the constraint $P(X_2 \geq 0) \geq \beta_2$ is binding. It is also true for the n-stage problem when none of the constraints $P(X_j \geq 0) \geq \beta_j$ is binding. Thus we know that $G_{sj}^* = 0$ in Theorem 6 without solving Problem 21 if Condition 22 is not satisfied.

Again, considering Problem 15, Theorem 8 can be proved.

__Theorem 8__

If the random variables $b_1, \ldots, b_n$ are independent, then a necessary condition that $\tilde{D}_\ell^{s*} = 0$ for some $\ell \in \Gamma^{s-1}$ is that $\tilde{F}_s^{-1}(0) > -\infty$ and either $F_s(k_\ell^{s-1}) = 0$ or $f_s\left[\tilde{F}_s^{-1}(0)\right] \geq f_s\left[F_s^{-1}(D^{\ell*})\right]$, where k_ℓ^{s-1} is defined in constraints 16 and 17.

This theorem holds for the case $n = 2$ even when the constraint $P(X_2 \geq 0) \geq \beta_2$ is binding. In RAC-TP-174[1], Theorems 7 and 8 are used to solve explicitly for the optimal decision rules of a particular two-period problem.

7. LINEAR PROGRAMMING UNDER UNCERTAINTY

A special case of Problem 1 that has been considered in the literature is the case in which $\alpha_i, \beta_j = 1, i, j = 1, \ldots, n$. Such problems have been named "linear programming under uncertainty."

The foregoing work gives the following theorem for this special case.

__Theorem 9__

Let $\alpha_i = 1, i = 1, \ldots, n$.
Let $\beta_j = 1, j = 1, \ldots, n$.
Then either $X_j^* = 0$, or

$$X_j^* = -\sum_{k=1}^{j-1} \frac{a_{jk}}{a_{jj}} X_k^* - \frac{\alpha_j}{a_{jj}} \cdot F_j^{-1}(0),$$

for all points $(b_1, \ldots, b_{j-1}) \in Q_{j-1}$.

Proof. From our definition of γ_ℓ^* we get $\gamma_\ell^* = 0$ for all ℓ as $\alpha_i = 1$ for all i implies that $1 - \alpha_i = 0$ for all i.

Therefore we must have $F_s\left[\bar{f}_s^{-1}(T_\ell^s)\right] = 0$ for all points in A_ℓ^{s-1} for $\ell \in I, J$.

Therefore $\bar{f}_s^{-1}(T_\ell^s) = \bar{F}_s^{-1}(0)$ for all points in A_ℓ^{s-1}, and hence the theorem is proved.

This result is particularly important because it illustrates dramatically the restriction of optimal action that occurs when the chance-constrained programming problem is restricted to a problem in linear programming under uncertainty. It should also be noted that the linear-programming-under-uncertainty problem has <u>no solution</u> for distributions (such as the normal distribution) for which $\bar{F}_s^{-1}(0) = -\infty$.

APPENDIXES

Appendix A

DERIVATION OF THE EULER EQUATION

Most texts of the calculus of variations derive the Euler equation for the problem
maximize

$$\int_a^b G\left(x, y(x), y'(x)\right) dx$$

subject to

$$y(a) = A \text{ and } y(b) = B \qquad (23)$$

In order to do this, they assume that in $[a,b]$ $y''(x)$ exists and is continuous and that all second partial derivatives of $G(.)$ exist and are continuous. They do not consider the case in which $G(.)$ is not a function of $y'(x)$ and so do not discuss what weaker conditions of continuity and differentiability of $G(x,y)$ are sufficient to obtain the Euler equation for this problem. Hence a derivation of the Euler equation for this special case is presented here.

Consider the problem
maximize

$$\int_a^b G\left(x, y(x)\right) dx , \qquad (24)$$

where it is assumed that $\partial G/\partial y$ exists and is continuous in $[a,b]$ and that $y(x)$ is continuous in $[a, b]$.

Let $J(y) = \int_a^b G[x, y(x)]\, dx$.

Let $\bar{y}(x)$ give a relative strong maximum to $J(y)$, i.e., $J(\bar{y}) \geq J(y)$ for all y such that $|y(x) - \bar{y}(x)| < \epsilon$ for all x in $[a, b]$ and some $\epsilon > 0$.

Let $y(x) = \bar{y}(x) + \epsilon f(x)$ be any other continuous curve such that $|y(x) - \bar{y}(x)| < \epsilon$ for all x in $[a, b]$.

Let $\phi(\epsilon) = J(\bar{y} + \epsilon f)$.

Then, since $\bar{y}$ is an extremum for $J(y)$, $d\phi(\epsilon)/d\epsilon\,|_{\epsilon=0} = 0$, [i.e., $\phi'(0) = 0$].

But $\phi(\epsilon) = \int_a^b G(x, \bar{y} + \epsilon f)\, dx$, so that $\phi'(0) = \int_a^b \left(\partial G(x,\bar{y})/\partial y\right) f(x)\, dx = 0$, which by the lemma of Lagrange (see Akhiezer[5])* implies that

$$\frac{\partial G(x, \bar{y})}{\partial y} = 0 \text{ for all } x \text{ in } [a, b] \qquad (25)$$

*See also Bateman[6] for a more complete discussion, including Haar's Lemma.

Hence Eq 25 is the Euler equation for Problem 24, and the existence and continuity of $\partial G/\partial y$ is a sufficient condition for the derivation of Eq 25.

The extension of Problem 24 to multiple integral isoperimetric problems can be achieved as it is in most texts of the calculus of variations. Hence Problem 5 requires that

$$\frac{d}{dX_s}\left[c_s X_s f_{s-1} + \lambda f_s\left(a_{ss} X_s + \sum_{i=1}^{s-1} a_{si} X_i + \alpha\right) f_{s-1}\right]$$

exist and be continuous in A_p^{s-1}. This is assured by the definition of A_p^{s-1}.

It is interesting to note that no end-point conditions exist on $y(x)$ in Problem 24 as in Problem 23. This is because the Euler equation (25) implicitly defines $y(x)$, and hence arbitrary end-point conditions would make the problem inconsistent. In the terminology of the calculus of variations there are the "natural conditions" at the end points in Problem 24. This is also the case in Problem 5.

Appendix B

AN ALTERNATIVE PROOF OF THEOREM 4

In Theorem 4 it is shown that the results of Theorem 2 could be extended to the complete n-period model if it is assumed that the constraints $P(X_j \geq \beta_j)$, $j = 1, \ldots, n$ are not binding. In this appendix a different and somewhat simpler approach is used to establish Theorem 4.

Problem 1 can be written in the form
maximize

$$\sum_{i=1}^{n} c_i \int \ldots \int X_i f_{i-1} \, db_1 \ldots db_{i-1}$$

subject to

$$\operatorname{sgn}(d_i) \int \ldots \int \bar{F}_i \left(- \sum_{j=1}^{i} a'_{ij} X_j - \omega'_i \right) f_{i-1} \, db_1 \ldots db_{i-1} \geq \alpha'_i, \quad i = 1, \ldots, n,$$

$$P(X_j \geq 0) \geq \beta_j, \quad j = 1, \ldots, n. \tag{26}$$

Now suppose that the constraints $P(X_j \geq 0) \geq \beta_j$, $j = 1, \ldots, n$ are not binding in Problem 26. Let u_i, $i = 1, \ldots n$ by

$$u_i(b_1, \ldots, b_{i-1}) = \sum_{j=1}^{i} a'_{ij} X_j + \omega'_i(b_1, \ldots, b_{i-1}), \quad i = 1, \ldots, n.$$

Then, by inverting these equations to get X_j as a function of u_i, $i = 1, \ldots, j$,

$$X_1 = - \frac{u_1}{a'_{11}} - \frac{\omega_1}{a_{11}}$$

$$X_2 = - \frac{u_2(b_1)}{a_{22}} - \frac{1}{a_{22}} \omega_2(b_1) - \frac{a_{21}}{a_{22}} X_1$$

$$= - \frac{u_2(b_1)}{a'_{22}} - \frac{a_{21}}{a_{22}} \left[- \frac{u_1}{a'_{11}} \right] \frac{a_{21}}{a_{22}} \frac{\omega_1}{a_{11}} - \frac{1}{a_{22}} \omega_2(b_1)$$

or, in general,

$$\lambda_i(b_1, \ldots, b_{i-1}) = \sum_{j=1}^{i} r_{ij} u_j(b_1, \ldots, b_{j-1}) + \sum_{j=1}^{i} \frac{r_{ij}}{d_j} \omega_j(b_1, \ldots, b_{j-1}) , \quad i = 1, \ldots, n, \tag{27}$$

where the $r_{ij}, i, j = 1, \ldots, n$ are constants that depend on the a_{ij} and d_j.

Putting Eq 27 into Problem 26 and ignoring the β_j constraints shows that Problem 26 is equivalent to
maximize

$$\sum_{i=1}^{n} c_i \int \ldots \int \left[\sum_{j=1}^{i} r_{ij} u_j + \sum_{j=1}^{i} \frac{r_{ij}}{d_j} \omega_j \right] f_{i-1} \, db_1 \ldots db_{i-1}$$

subject to

$$\operatorname{sgn}(d_i) \int \ldots \int \bar{F}_i \left(u_i(b_1, \ldots, b_{i-1}) \right) f_{i-1} \, db_1 \ldots db_{i-1} \; \alpha_i' , \; i = 1, \ldots, n. \tag{28}$$

This is equivalent to

$$\int \ldots \int \sum_{i=1}^{n} \sum_{j=1}^{i} c_i \frac{r_{ij}}{d_j} \omega_j f_{i-1} \, db_1 \ldots db_{j-1} +$$

maximize

$$\sum_{j=1}^{n} \int \ldots \int \sum_{i=j}^{n} c_i r_{ij} u_j f_{j-1} \, db_1 \ldots db_{j-1}$$

subject to

$$\operatorname{sgn}(d_j) \int \ldots \int \bar{F}_j(u_j) f_{j-1} \, db_1 \ldots db_{j-1} \geq \alpha_j' , \; j = 1, \ldots, n. \tag{29}$$

In transforming Problem 28 into Problem 29, the region of integration was changed from $\bar{Q}_{i-1}$ to $\bar{Q}_{j-1}$. This was done by first observing that in the objective function of Problem 28 u_j is being integrated, and, in our enumeration, $i \geq j$. Now if $i > j$, the term $c_i r_{ij} u_j (b_1, \ldots, b_{j-1})$ can be factored outside the integral sign, and the integration of f_{i-1} can be performed with respect to $b_j, b_{j+1}, \ldots, b_{i-1}$. This means that integration is being performed over all possible values of these random variables. Hence the value of this integration is 1, and integration must be performed over $\bar{Q}_{j-1}$.

However, Problem 29 is now separable into the following n distinct and unrelated problems of determining $u_j, j = 1, \ldots, n$, viz,
maximize

$$\int \ldots \int \left(\sum_{i=j}^{n} c_i r_{ij} \right) u_j f_{j-1} \, db_1 \ldots db_{j-1}$$

subject to

$$\operatorname{sgn}(d_j) \int \ldots \int \bar{F}_j(u_j) f_{j-1} \, db_1 \ldots db_{j-1} \geq \hat{\alpha}_j . \tag{30}$$

Since Problem 30 is a special case of Problem 4, we can proceed to solve Problem 30 just as Problem 4 was solved. Establish that a necessary condition that u_j^* maximize Problem 30 is that there exist a covering of $\bar{Q}_{j-1}$, say $\{A_\ell^{j-1}, \ell \in \mathcal{L}^{j-1}\}$, such that $u_j^* = \bar{F}_j^{-1}(T_\ell^{j*})$ in A_ℓ^{j-1}, $\ell \in \mathcal{L}^{j-1}$, where T_ℓ^{j*} can have only three possible values: V_ℓ^j, W_ℓ^j, or T^{j*}.

If it is now assumed that the random variables are independent, D_ℓ^j can be defined as was done in Sec 6 and the optimal D^j and A_ℓ^j can be found as was outlined in the development preceding Theorem 6. Thus u_j^* is determined.

Since this can be done for each j, $j = 1, \ldots, n$ in Problem 30, u_j^*, $j = 1, \ldots, n$ can be found. Substituting these expressions into Eq 27,

$$X_i^* = \sum_{j=1}^{i} r_{ij} u_j^* + \sum_{j=1}^{i} \frac{r_{ij}}{d_j} \omega_j$$

$$= - \sum_{j=1}^{i-1} \frac{a_{ij}}{a_{ii}} X_j^* - \frac{1}{a_{ii}} \omega_i + \tilde{F}_i^{-1} (D_\ell^{i*}) \text{ in } A_\ell^{i-1},$$

which agrees with the previous results.

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